1. Vertical beam size & 2. Nonlinear Dynamics

- Emittance and equilibrium beam size
- Resonances
- Coupling correction
- Nonlinear dynamics
Individual particles execute betatron oscillations about the beam center:

\[ x_\beta(s) = \sqrt{\varepsilon_A} \sqrt{\beta_x(s)} \cos(\varphi_x(s) + \varphi_0) \]

where \( \beta_x(s) \) is an amplitude that varies with position around the ring, and \( \varepsilon_A \) is a constant.

The electrons in an electron bunch are distributed equally in betatron phase and have a Gaussian transverse distribution.

The horizontal emittance specifies the width of this Gaussian:

\[ \varepsilon_x = \sigma_x^2 / \beta_x(s) \]

The emittance is determined by the balance between damping and excitation of oscillations from synchrotron radiation.
Radiation damping & Quantum excitation

- **Radiation damping:**
  - Longitudinal & transverse momentum is lost to synchrotron radiation
  - Only longitudinal momentum replaced by RF cavity

- **Quantum excitation:**
  - A photon radiated at some place with dispersion shifts the closed orbit for the electron.
  \[ \Delta x_{c.o.} = -\eta_x \varepsilon_p / E \]
  - Electron oscillates about new closed orbit: excitation of betatron oscillation.
Emittance and beam size

- Balance between damping and quantum excitation leads to an equilibrium emittance:

\[ \varepsilon_x \propto \int_{dipoles} ds(\gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta_x' + \beta_x \eta_x'^2) \]  

(1)

- The total horizontal beam size has a contribution from \( \beta \) oscillations as well as a contribution from energy spread:

\[ \sigma_x = \sqrt{\beta_x \varepsilon_x + \eta_x^2 \left( \frac{\sigma_E}{E} \right)^2} \] 

(2)

- In ideal accelerator \( \eta_y = 0 \), so \( \sigma_y \) is negligibly small.

- In reality, vertical beam size has 3 contributions:
  - Quantum excitation of \( y_\beta \)-oscillations as in eqn. 1 above (all \( x \rightarrow y \)) through small \( \eta_y \).
  - Orbit spread due to \( \sigma_E \eta_y \) as in eqn. 2 above.
  - Coupling of \( x_\beta \)-oscillations into \( y_\beta \).
Review

So far we've looked at three solutions to eqn. of motion:

1. \[ x'' + K_x(s)x = 0 \] \[ x_\beta(s) = \sqrt{\varepsilon_A} \beta(s) \cos(\varphi(s) - \varphi_0) \] \[ \beta \text{ oscillations} \]

2. \[ y'' + K_y(s)y = \theta \delta(s - s_0) \] \[ y_{c.o.}(s) = \theta \frac{\sqrt{\beta(s)\beta(s_0)}}{2 \sin(\pi \nu)} \cos(|\varphi(s) - \varphi(s_0)| - \pi \nu) \] \[ \text{Closed orbit shift} \]

3. \[ x'' + K_x(s)x = \frac{\delta_\varepsilon}{\rho} \]
\[ x(s) = \eta \delta_\varepsilon \] \[ \text{Dispersion} \]

Now a fourth:

\[ x'' + K_x(s)x = -K_s(s)y \]
\[ y'' + K_y(s)y = -K_s(s)x \] \[ \text{Coupling} \]

\[ (\text{where } K_x = -K_y(s) = -\frac{1}{B\rho} \frac{\partial B_y}{\partial x} \quad K_s(s) = \frac{1}{B\rho} \frac{\partial B_x}{\partial x}) \]
Difference coupling resonance (approximate math)

- The equation \( y'' + K_y(s)y = -K_s(s)x \) is like a driven harmonic oscillator.

- If a harmonic oscillator is driven on resonance, get large oscillations – large vertical beam size, in this case.

- On resonance if vertical drive is at the vertical tune:
  \[
  K_s(s)x \sim \exp\left(i2\pi v_y \frac{s}{L_0}\right)
  \]
  i.e. \( K_s(s) \sim \exp\left(i2\pi(v_y - v_x) \frac{s}{L_0}\right) \)

- But \( K_s(s) \) is periodic in \( L_0 \) (the ring circumference), so it only has Fourier terms period in \( L_0 \):
  \[
  K_s(s) \sim \exp\left(i2\pi n \frac{s}{L_0}\right), \quad n = \text{integer}
  \]

- So \( y_\beta \) driven on resonance if:
  1. \( v_x - v_y = n \) (integer) this is a difference coupling resonance
  2. \( K_s(s) \) has a spatial Fourier component at \( n \) to drive this resonance
More resonances

- Skew quadrupoles, $K_s(s)$, can also drive the sum coupling resonance, if:
  1. $\nu_x + \nu_y = n$ (integer)
  2. $K_s(s)$ has a spatial Fourier component at $n$ to drive this resonance

- Skew quadrupoles can also couple $\eta_x$ to create $\eta_y$.

\[ y'' + K_y(s)y = -K_s(s)(x_\beta + \eta_x \delta_\varepsilon), \quad K_s \eta_x \delta_\varepsilon \text{ drives } \eta_y \]

- This analysis extends to nonlinear driving terms

\[ x'' + K_x(s)x = A_{i,j} x^i y^j, \quad y'' + K_y(s)y = B_{i,j} x^i y^j \]

- Generating nonlinear resonances, when:

\[ n \nu_x + m \nu_y = N \quad (n, m, N \text{ integers}) \]
Skew quadrupole corrector distribution

When choosing where to locate skew quadrupoles, must distribute them to correct both phases of coupling resonances and vertical dispersion:

- Distribute in difference coupling resonance phase
  \[
  \kappa = \frac{1}{4} \pi \int ds K_s \sqrt{\beta_x \beta_y} e^{i\Phi_D} \quad \frac{\Phi_D(s)}{2\pi} = (\mu_x(s) - \mu_y(s)) - \frac{s}{C}(\nu_x - \nu_y - N)
  \]

- In sum coupling resonance phase
  \[
  \frac{1}{4} \pi \int ds K_s \sqrt{\beta_x \beta_y} e^{i\Phi_S} \quad \frac{\Phi_S(s)}{2\pi} = (\mu_x(s) + \mu_y(s)) - \frac{s}{C}(\nu_x + \nu_y - M)
  \]

- And in \( \eta_y \) phase
  \[
  \eta_x \sqrt{\beta_y} e^{i\Phi_{\eta_y}} \quad \frac{\Phi_{\eta_y}(s)}{2\pi} = \mu_y(s) - \frac{s}{C}(\nu_y - N)
  \]

- Need some skew quadrupoles at non-zero \( \eta_x \)
Resonance Description of Global Coupling

- Global coupling is typically described using a resonance theory
- Difference coupling resonance
  \[ \kappa = \frac{1}{4\pi} \int ds K_s \sqrt{\beta_x \beta_y} e^{i\phi_D} \]
  \[ \frac{\phi_D}{2\pi} = \mu_x(s) - \mu_y(s) - \frac{s}{C} \Delta_r \]
  \[ \Delta_r = (\nu_x - \nu_y - N) \]
- Vertical emittance near difference resonance:
  \[ \frac{\varepsilon_y}{\varepsilon_x} = \frac{|\kappa|^2}{|\kappa|^2 + \Delta_r^2 / 2} \]

\( \kappa \) is resonance strength, \( \Delta_r \) is distance from resonance.
Scan of difference resonance

- There are sum resonances as well (phase advance proportional to sum of horizontal and vertical phase advance) and of course higher order resonances.
- One can create orthogonal knobs of skew quadrupoles directly acting on one of those coupling resonances.

Minimum tune split (on resonance):

\[
(\nu_x - \nu_y)_{\text{min}} = 2|\kappa|
\]
More advanced coupling correction algorithms

- Using digitized turn-by-turn driven oscillations (CESR)
  - with no coupling, if you drive the horizontal tune, you see only horizontal oscillations
  - with coupling, see some vertical oscillation as well.
  - measure vertical oscillations at all BPMs; adjust skew quadrupoles to minimize them.

- Using closed orbit response – LOCO (NSLS, ESRF, ALS, SPEAR3)
  - Correcting closed orbit coupling not the same as correcting betatron coupling, but simulations and experiments show it’s close enough.
  - With LOCO, can simultaneously correct $\eta_y$ to get minimum $\sigma_y$. 
Coupling & $\eta_y$ correction, LOCO

Minimize $\eta_y$ and off-diagonal response matrix:

Lifetime, 19 mA, single bunch

- Correction off: 4.5 hours
- Correction on: 1.5 hours
Simulation of coupling correction with LOCO

- Use accelerator toolbox (Andrei Terebilo), Matlab and LOCO (James Safranek, Greg Portman) for simulations
- Use random skew error seeds
- Try to find effective skew corrector distributions and to optimize correction technique in simulation
- Used two correction approaches:
  1. Response Matrix fitting – ‘deterministic’, small number of iterations
  2. Direct minimization (nelder-simplex, …) – easy to do on the model, difficult on real machine
    - Surprisingly both approaches gave about the same performance in the model calculations
    - For response matrix analysis you have to optimize several parameters of the code as well (weight of dispersion, number of SVs, use of effective model/full model …)

Thanks Christoph
Weight of dispersion in LOCO fit

- The relative contribution of vertical dispersion and coupling to the vertical emittance depends on the particular lattice (and the particular error distribution).
- Therefore the optimum weight for the dispersion in the LOCO fit has to be determined (experimentally or in simulations).
- The larger the weight factor, the better the vertical dispersion gets corrected, but eventually the coupling ‘explodes’.
- Set weight to optimum somewhat below that point.
- Outlier rejection tolerance might be important parameter as well.
Finding an Effective Skew Quadrupole Set

- To find an effective skew quadrupole distribution, we used several correction methods, first in simulations – best method was orbit response matrix fitting (using LOCO).
- Predictive method, can be easily used on real machine.
- Issues are:
  - Cover set of phases relative to dominant coupling resonance(s)
  - Magnets should be distributed around the ring in order to avoid excessive local coupling/vertical dispersion
  - Need different values of dispersion/beta function to be effective both for coupling and vertical dispersion correction
- Set of 12 skew quadrupoles was reasonably efficient
LOCO coupling correction ALS/NSLS

- Achieved an emittance reduction from 150 pm (routine ALS operation) to about 5 pm (pictures on the right illustrate size reduction for insertion device straights)

- This was a world record in 2003 and about the NLC damping ring design value

- Correspondingly the brightness increases by factor 30 (for hard x-rays – because of diffraction limit less for soft x-rays)
Further reading on coupling correction

- Guignard, CERN 76-06 1976
- (De Ninno & Fanelli, PRST-AB, Vol 3, 2000).
- K. Ohmi et al., PRE 49, No 1, 1994
- D. Sagan and D. Rubin, PRST-AB, Vol 2, 1999
- C. Steier, and D. Robin, EPAC’00.
- P. Nghiem, and Tordeux, Coupling correction for the ESRF, SOLEIL internal report, 1999.
- R. Nagaoka, EPAC’00.
- R. Nagaoka, and L. Farvacque, PAC’01.
- C. Steier, et al., ‘Coupling Correction and …’, PAC 2003
Nonlinear dynamics - motivation

Motion of particles at large amplitudes impacts the performance of the storage ring

Particle loss:

- Injection efficiency
  - Longer injection times
  - Increased radiation levels

- Lifetime
  - More frequent fills
  - Faster current loss (varying brightness/photon optics thermal load)
Injection efficiency

Storage ring dynamic aperture must be large enough to capture sufficient fraction of injected beam.
Storage ring lifetime

- **Elastic Scattering**
  \[ \frac{1}{\tau_{el}} \propto \frac{1}{E^2} \times \left( \frac{\beta_x}{\Delta_x^2} \langle p \beta_x \rangle + \frac{\beta_y}{\Delta_y^2} \langle p \beta_y \rangle \right) \]  

- **Touschek Effect**
  \[ \frac{1}{\tau_{tou}} \propto \frac{1}{E^3} \frac{I_{bunch}}{V_{bunch}} \frac{1}{\sigma_x} f(\epsilon, \sigma_x', E) \]  

- **Quantum Lifetime**
  \[ \frac{1}{\tau_q} \propto \frac{\Delta^2}{\sigma^2} \times \exp\left(-\frac{\Delta^2}{2\sigma^2}\right) \]  

- **Inelastic Scattering**
  \[ \frac{1}{\tau_{inel}} \propto \langle p \rangle \times \ln(\epsilon) \]  

\[ \frac{1}{\tau} = \frac{1}{\tau_{el}} + \frac{1}{\tau_{tou}} + \frac{1}{\tau_{ql}} + \frac{1}{\tau_{inel}} \]
Resonances

- Nonlinear driving terms in the equation of motion

\[ x'' + K_x(s)x = A_{i,j}x^i y^j, \quad y'' + K_y(s)y = B_{i,j}x^i y^j \]

- Generate nonlinear resonances, when:
  1. \( n \nu_x + m \nu_y = N \) \( (n, m, N \text{ integers}) \)
  2. \( A_{ij} \) has the \( N \text{th} \) spatial Fourier component to drive this resonance

- Benefit of periodicity, resonance reduction:

Tune plane:
- Resonances to 5th order
- 12-fold periodicity =\( x_{12} \)
- Only get resonances \( n \nu_x + m \nu_y = 12 \times N \)
- True for periodic magnets (sextupoles), not IDs
Resonance excitation

- Resonances can lead to irregular and chaotic behavior in betatron oscillations, which can lead to particle loss.

  Rule of thumb ---- Avoid low order resonances

- Unfortunately there is no simple way to forecast the real strength of resonances without using a tracking code or through measurements

  - Tune scans
  - Frequency map analysis
Dynamic aperture vs. tune

- Resonant lines:
  - $\nu_x - \nu_y = 9$
  - $3\nu_x + \nu_y = 48$
  - $4\nu_x + \nu_y = 62$

- Resonances offset from tune shift with amplitude.

- * = operating tunes (14.19, 5.23)

- Data gathered automatically on owl shift.
At BESSY, the beam loss was measured as a function of tunes. The additional losses associated with an insertion device showed a problem with nonlinear fields.

Kuske et al., PAC01.
ALS tune scans (with and w/out $\beta$-beating)

Three resonances are present:

\[
5\nu_x = 72 \quad \text{(allowed)}
\]

\[
3\nu_x = 43 \quad \text{(unallowed)}
\]

\[
2\nu - \nu = 37 \quad \text{(unallowed)}
\]
ALS computer tracking results show benefit of correcting periodicity of linear optics.

Near 3rd order resonance

Near 5th order resonance

linear optics corrected

linear optics uncorrected
Tune scan summary

Advantages

Quickly and sensitively see excited resonances in the tails and core of the beam as a function of different tunes

Disadvantages

Probing different machines and not looking at the effect of resonances on one working point and at different amplitudes. This is what one really would like to see.
Tune shift with amplitude

- Electron tunes get shifted with $x_\beta$, $y_\beta$ oscillation amplitude.
- Tune shift with square of amplitude ($x_\beta^2$) comes from cubic terms in equation of motion, i.e.

$$x'' + K_x(s)x = A_3x^3$$
Frequency map analysis

Frequency Space

Amplitude Space

Color scheme refers to diffusion rate of tunes.
Frequency maps – ideal lattice vs. lattice with small linear optics errors.
Frequency maps – experimental set-up

Experimental Hardware

- horizontal + vertical single turn kicker
- 96 turn by turn monitors (1024 turns)

Experimental Procedure

- Electron beam (single bunch or small bunch train) gets simultaneously a horizontal and a vertical kick
- Beam centroid oscillations are recorded turn by turn for 1024 turns
- Repeat with different initial conditions (hor. + vert. kick amplitude) → 400-600 total points per map

Data Analysis

- turn by turn data is analyzed with frequency analysis post processor (NAFF) and results plotted in tune plane
Measured frequency maps at ALS

- excellent agreement, using calibrated model (gradient errors), random skew errors, nominal sextupoles

\[ \text{Phys. Rev. Lett. 85, 3, (July 2000), pp.558-561} \]

Also see PAC papers for good measurement results at BESSY-II.
Computer tracking example

Vertical amplitude growth due to resonances:

Frequency Space

Amplitude Space
Measuring nonlinear dynamics - summary

1. Tune scans
2. Tune maps
3. Closed orbit scans (see tomorrow's lecture)